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Dynamics of the Current-Driven Josephson Junction

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The irreversible macroscopic dynamics of the Josephson junction coupled to external wires acting as a current source is derived rigorously from the underlying microscopic Hamiltonian quantum mechanics. The external systems are treated in the singular coupling limit. The use of this limit is explicitly justified via an interpretation of the singular coupling limit in terms of the relative magnitudes of system, reservoir, and coupling energies. The qualitative behavior of the macroscopic dynamical equations is shown to depend sensitively and crucially on the interaction between the wires and the superconductors and on the size of the wires: the dc Josephson effect only happens when one lets Cooper pairs be driven into the junction by collective (i.e., "small") reservoirs.

KEY WORDS: Josephson junction; strong coupling BCS model; mean-field Hamiltonian; open system; singular coupling limit; quantum dynamical semigroup; individual and collective reservoirs; extremal permutation-invariant states; macroscopic dynamics, dc Josephson effect.

1. INTRODUCTION

The rigorous theory of open quantum mechanical systems is not very old. It originates essentially from a classical paper by Hepp and Lieb,[(1)] who treated the coupling to the external quantum systems ("reservoirs") by introducing the so-called "singular coupling limit," and the equally well-known work of Davies,[(2)] who showed that the so-called "weak coupling limit" can be performed in a mathematically exact way. It was shown later[(3)] that both types of couplings can be treated on the same footing: one uses the general master equation (constructed first by Nakajima, Zwanzig, and others) and shows that under two different limiting conditions the memory terms therein can be neglected. The ensuing semigroup structure

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of the dynamics of the open system motivated detailed investigations of quantum dynamical semigroups and their generators.[4–6] The theory is thus well established (for reviews, see refs. 7 and 8).

However, while especially the weak coupling limit has been thoroughly studied for finite open systems,[9] for the potentially richer infinite-volume case there seem to be only very few explicit model discussions, applying the techniques to concrete physical systems. I know of studies of the Ising model,[10] the BCS model,[11] and the Bose gas[12,13] in the weak coupling limit, and of the laser[14] in the singular coupling limit. The motivation for such studies is at least twofold. First, they give a rigorous microscopic foundation for the visible behavior of the respective macroscopic systems; on the other hand, they exemplify and can reveal more fully the physical nature of the limiting situations. I emphasize this last point, as it is not often discussed explicitly in the literature; in particular, the fact that the two limits are mathematically very similar[15] should not obscure the fact that they are utterly different physically (compare ref. 16).

In this paper I study the Josephson junction as an open system in the sense that it is driven by external wires acting as a current source. My principal aim is to provide a rigorous model for the dc Josephson effect. The dc Josephson effect has acquired great importance lately in the study of the so-called "macroscopic quantum phenomena"[17,18] and a thorough theoretical clarification of this conceptually difficult area seems to require an account of the underlying systems which is as complete and exact as possible. The dc currents in the junction necessitate connecting it to current-carrying wires. I treat these external systems in (a slightly generalized form of) the singular coupling limit, arguing that this limit is the physically proper choice as compared to the weak coupling limit, which has also been used in similar situations.[19,20] Such arguments, however, presuppose a closer examination of the singular coupling limit and its physical interpretation, which needs to be contrasted to the one of the weak coupling limit. Consequently, I devote a seperate section (Section 3) to these questions, where I stress that both limits can be justified by (and hence interpreted as the reflection of) specific relations among the system, the reservoir, and the coupling energies.

Besides the choice of the appropriate coupling limit, a model of the external wires requires further physical considerations: one can take individual reservoirs (i.e., one for each microscopic degree of freedom of the system) or collective ones, and one can have electrons or Cooper pairs being fed into the junction. The various possible choices can differ widely with respect to the ensuing macroscopic dynamical equations. Indeed, while I shall let Cooper pairs flow into the system out of a collective reservoir and arrive at a complete description of the dc Josephson effect, a model with electrons out of individual reservoirs, briefly considered in ref. 1, leads to different equations exhibiting rather unphysical behavior. This sheds some light on the fact that in the singular coupling limit, the dynamics is very sensitive even qualitatively to the nature of the coupling, in contrast to the situation in the weak coupling limit.[11]

Having thus given an overview on the conceptual problems to be dealt with here, I now briefly describe the mathematical route I have taken. In Section 2 I recall the microscopic, mean-field model Hamiltonian for the closed Josephson junction developed in previous work[21] and establish the ensuing equations of motion for the expectation values of the physically relevant intensive observables in the infinite volume limit.[1] They are valid in a large class of nonequilibrium states and form an autonomous set of nonlinear differential equations, contracting the description of the junction to the evolution of the Cooper pair density, the electron number difference, and the phase difference between the two superconductors. The treatment of the open system is prepared in Section 3 with an interpretative discussion of the coupling limits. In Section 4 I construct the wires and their interaction with the superconductors and calculate the generator of the semigroup arising from this coupling for the finite junction dynamics in the singular coupling limit. A theorem of Ito and Nakagomi[22] makes it possible to derive from there the equations of motion for the infinite open system in much the same way as for the closed system. These equations exhibit the dc Josephson effect and are shown to contain the phenomenological theory as an approximation.

In this way, the present treatment is very similar to the procedure in refs. 1 and 10–12, in particular with regard to the order of the various limits taken: first, the external systems are treated in the thermodynamic limit; second, the singular coupling limit is performed to obtain a quantum dynamical semigroup for the evolution of the finite Josephson junction. Finally, I take the infinite-volume limit of the junction itself—it is only then that the dynamics can be conclusively analyzed.

The main difference from the approach of Hepp and Lieb is that they work with Langevin equations for the time-dependent system operators, while I use the master equation, describing the reservoir influence with semigroup generators. The latter approach is more flexible (the present model cannot be solved via Langevin equations) and makes proofs easier; the former contains more information about fluctuations, which, however, I do not consider here. An interesting question which I have not followed up is whether the model could also be treated with the methods developed in ref. 23.

Finally, I compare this work with previous treatments of the open

Josephson junction. Essentially the same model of the wires has been considered in refs. 24 and 25 with structurally equivalent results, but the derivation is not rigorous. The model of the current driven junction due to Hepp and Lieb,[1] which was also used in ref. 26, has been already mentioned; these treatments are rigorous, but one can show (Section 4.4) that the ensuing macroscopic dynamics does not give a physically satisfying account of the junction behavior.

As a last comment, I mention that the model structures developed in this paper could equally well be used to describe the Fiske effect[27] in the Josephson junction: it poses no particular problem to treat a coupling of the system to electromagnetic modes in the junction cavity with the same methods (see ref. 26).

2. DYNAMICS OF THE CLOSED JOSEPHSON JUNCTION

2.1. The Model Hamiltonian

I use the quasispin formulation of the strong coupling BCS model for the two superconductors R and S constituting the junction, which I briefly review. The operators (for R)

$$\sigma_{\mathbf{k}}^{+},\ \sigma_{\mathbf{k}}^{-},\ \sigma_{\mathbf{k}}^{3} \tag{1}$$

create, annihilate, and count the electrons in pairs $(\mathbf{k}, \uparrow; -\mathbf{k}, \downarrow)$; here, $\mathbf{k}$ runs over all momenta whose associated energies $\varepsilon(\mathbf{k})$ lie within a finite region around the Fermi surface μ_R of R, assumed to be cubic with volume a^3:

$$\mathbf{k} \in \Lambda_R(a) := \left\{ \mathbf{k} = \frac{2\pi \mathbf{m}}{a},\ \mathbf{m} \in Z^3,\ |\varepsilon(\mathbf{k}) - \mu_R| \leqslant \hbar\omega_D \right\} \tag{2}$$

Let $\{1, 2, 3, \ldots, N\}$ be some denumeration of $\Lambda_R(a)$; the algebra of operators for the finite superconductor is then defined as

$$\mathscr{A}_R(N) := \bigotimes_{n=1}^{N} \mathscr{A}_n, \qquad \mathscr{A}_n := lin_{\mathbb{C}} \{\sigma_n^+, \sigma_n^-, \sigma_n^3, 1_n\} \tag{3}$$

The operators σ_n^i $(i = +, -, 3)$ obey spin commutation relations and can therefore be represented by Pauli matrices on the Hilbert space $\mathbb{C}^2$:

$$\sigma_n^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_n^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_n^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

For the infinite system we have

$$\mathbf{k} \in \Lambda_R(\infty) := \left\{ \mathbf{k} = \frac{2\pi \mathbf{m}}{na}, \mathbf{m} \in Z^3, n \in \mathbb{N}, |\varepsilon(\mathbf{k}) - \mu_R| \leqslant \hbar\omega_D \right\} \tag{5}$$

and the operator algebra is

$$\mathscr{A}_R := \bigotimes_{n \in \mathbb{N}} \mathscr{A}_n \tag{6}$$

which is naturally endowed with a quasilocal structure. The physically important condensed pairs (Cooper pairs) are described with the intensive observables

$$r_N^{\pm} := \frac{1}{N} \sum_{n=1}^{N} \sigma_n^{\pm} \tag{7a}$$

so that $c_N^R := r_N^+ r_N^-$ is the observable for the density of Cooper pairs in R.[21] The operator

$$r_N^3 = \frac{1}{N} \sum_{n=1}^{N} \sigma_n^3 \tag{7b}$$

measures the particle density in Λ_R in a convenient normalization; it is related to the number operator N_R in the following way:

$$N_R = 2Nr_N^3 + N1$$

We can now write down the BCS Hamiltonian:

$$H_{\mathrm{BCS}}^R(N) = N(\varepsilon_R(2r_N^3 + 1_N) - g r_N^+ r_N^-) \tag{8a}$$

The same structures can be established for the right superconductor S; this leads to the operators s_N^i, $i = +, -, 3$, and the Hamiltonian

$$H_{\mathrm{BCS}}^S(N) = N(\varepsilon_S(2s_N^3 + 1_N) - g s_N^+ s_N^-) \tag{8b}$$

Accordingly, we have

$$\mathscr{A} := \mathscr{A}_{\mathrm{system}} := \mathscr{A}_R \otimes \mathscr{A}_S = \bigotimes_{n \in \mathbb{N}} (\mathscr{A}_n^R \otimes \mathscr{A}_n^S) \tag{9}$$

as the operator algebra for the Josephson junction.

The coupling between R and S consists of a tunneling Hamiltonian

$$H_T(N) = N\lambda(r_N^+ s_N^- + s_N^+ r_N^-) =: Nw_N \tag{10}$$

describing the tunneling of condensed pairs, and an electrostatic interaction part, stemming from the capacity of the junction. I have argued elsewhere[21] that this electrostatic energy should be taken into account in this framework with a Hamiltonian

$$H_C(N) = N \cdot 2Kz_N^2; \qquad K = \frac{e^2}{C}, \qquad z_N := \frac{1}{2}(r_N^3 - s_N^3) \tag{11}$$

describing charge imbalances due to nonequilibrium distributions within Λ_R, Λ_S, and by setting (in general)

$$\varepsilon_R - \varepsilon_S =: \Delta\varepsilon = \mu_R - \mu_S \neq 0 \tag{12}$$

in order to cover the charge imbalance due to different values of μ (different positions of the Fermi surface). These are brought about by a fixed dc voltage across the junction. One then has (z denotes the expectation value of z_N)

$$V = V^{\mathrm{DC}} + V^{\mathrm{AC}} = \frac{\Delta\varepsilon}{e} + \frac{2K}{e} z \tag{13}$$

as the expression for the total voltage.

In this way, the model Hamiltonian for the closed Josephson junction becomes

$$\begin{aligned} H_{\text{system}}(N) := N\{&(\varepsilon_R + \varepsilon_S)\, 1_N + 2(\varepsilon_R r_N^3 + \varepsilon_S s_N^3) - g(r_N^+ r_N^- + s_N^+ s_N^-) \\ &+ \lambda(r_N^+ s_N^- + s_N^+ r_N^-) + 2Kz_N^2\} \end{aligned} \tag{14}$$

All constants are chosen in such a way that $H_{\text{system}}(N)$ is of mean-field type and strictly extensive $[O(N)]$.

2.2. Dynamics of Intensive Observables

The structures defined so far are easily seen to fall within the general setup discussed by Hepp and Lieb in ref. 1. In order to consider the macroscopic dynamical equations of the model (i.e., the evolution of expectation values of intensive observables in the thermodynamic limit), let us introduce the operator describing the pair current,

$$j_N := (-i)\, \lambda(r_N^+ s_N^- - s_N^+ r_N^-) \tag{15}$$

One then has as a set of physical operators which form a closed set under the commutator $N[\ ,\ ]$

$$\begin{aligned} r_N^3, s_N^3 &= \text{particle densities of R and S} \\ c_N^R, c_N^S &= \text{Cooper pair densities of R and S} \\ w_N &= \text{barrier energy (density)} \\ j_N &= \text{Josephson current (density)} \end{aligned} \tag{16}$$

The Heisenberg equations of motion for these operators are $\{(d/dt)\, a_N = i[H_{\text{system}}(N), a_N]\}$

$$\begin{aligned} \frac{d}{dt} r_N^3 &= j_N \\ \frac{d}{dt} s_N^3 &= -j_N \\ \frac{d}{dt} c_N^R &= -2j_N r_N^3 \\ \frac{d}{dt} c_N^S &= 2j_N s_N^3 \\ \frac{d}{dt} w_N &= -2\Delta\varepsilon\, j_N - (4K+4g)\, j_N z_N \\ \frac{d}{dt} j_N &= 2\Delta\varepsilon\, \omega_N + (4K+4g)\, w_N z_N - 4\lambda^2(c_N^R s_N^3 - c_N^S r_N^3) \end{aligned} \tag{17}$$

If we now restrict ourselves to extremal permutation-invariant states $\omega \in \partial S^P(\mathscr{A})$ [which are just the product states $\omega = \bigotimes_{n \in \mathbb{N}} \rho$, $\rho \in S(\mathscr{A}_n^R \otimes \mathscr{A}_n^S)$; for example, the BCS ground states are such "pure classical"[(1)] or "macroscopic"[(11)] states], we can immediately apply Theorem 2.3 of ref. 1 to obtain the dynamics of expectation values of the operators (16) in the thermodynamic limit $N \to \infty$: they obey the very same equations (17), thus forming a classical, nonlinear set of differential equations.

Denoting these expectation values by simply dropping the N index, it is easy to show that under the physically natural choices

$$r^3 + s^3 \equiv 0, \qquad c^R = c^S =: c \quad \text{at } t = 0 \tag{18}$$

(which are fulfilled, e.g., for the BCS ground states), the system (17) simplifies to

$$\begin{aligned}
&\frac{d}{dt} z(t) = j(t)\\
&\frac{d}{dt} c(t) = -2j(t)\, z(t)\\
&\frac{d}{dt} w(t) = -2\Delta\varepsilon\, j(t) - (4K+4g)\, j(t)\, z(t)\\
&\frac{d}{dt} j(t) = 2\Delta\varepsilon\, w(t) + (4K+4g)\, w(t)\, z(t) - 8\lambda^2 c(t)\, z(t)
\end{aligned} \tag{19}$$

Barrier energy w and Josephson current j are related through

$$\Delta\varphi = \varphi_R - \varphi_S \tag{20}$$

where φ_R (φ_S) is the phase angle of the superconductor R(S). Explicitly,

$$\begin{aligned}
j(t) &= 2\lambda c(t) \sin \Delta\varphi(t)\\
w(t) &= 2\lambda c(t) \cos \Delta\varphi(t)
\end{aligned} \tag{21}$$

These equations follow directly from the definitions and the relations

$$r^+(t) = [c(t)]^{1/2}\, e^{i\varphi_R(t)}, \qquad s^+(t) = [c(t)]^{1/2}\, e^{i\varphi_S(t)}$$

As the final equations of motion for the closed Josephson junction, written in the dynamical variables

$$\begin{aligned}
\Delta\varphi &= \text{phase difference between R and S}\\
z &= \text{difference of particle densities (within } \Lambda)\\
c &= \text{Cooper pair density}
\end{aligned} \tag{22}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}(c + z^2) = 0\\
&\frac{d}{dt} z = j = 2\lambda c \sin \Delta\varphi\\
&\frac{d}{dt} \Delta\varphi = 2\Delta\varepsilon + (4K+4g) z - 4\lambda \cos \Delta\varphi\, z
\end{aligned} \tag{23}$$

The physical content of these equations, exhibiting several additional terms as compared to the classical Josephson relations, is discussed in ref. 21; for present purposes, it suffices to point out that these equations do not describe the dc Josephson effect: $j = \text{const} \neq 0$ entails $z = \text{const} \cdot t$, giving $\Delta\varphi \neq \text{const}$ and hence $j \neq \text{const}$. This is not surprising, since in a closed system of this kind there can be no permanent dc currents. To describe such situations consistently, it is imperative to take into account an external current source, i.e., to treat the junction as an open system. This treatment is prepared in the next section.

3. THE SINGULAR COUPLING LIMIT AND ITS INTERPRETATION

In order to discuss the physical meaning of the singular coupling limit, I first briefly review its mathematical structure; for definiteness and in view of the present applications, I shall be more specific than is necessary (for a more general and complete review, see ref. 4).

Consider an open system, described by a finite Hilbert space $\mathscr{H}_S$, creation and annihilation operators $a_1^{\#}, \ldots, a_N^{\#}$, and a Hamiltonian H_S; it is coupled to a reservoir consisting of one or more identical, infinite, and quasifree Fermi systems. Each of them is described by the operator algebra $\mathrm{CAR}(L^2(I, d\omega))$, where $I \subseteq R$ is the spectrum $\sigma(h)$ of the one-particle Hamiltonian

$$h\colon\ L^2(I) \to L^2(I)$$
$$f(\omega) \mapsto \omega f(\omega)$$

Hence, I describes the energy modes of the reservoir. We take each reservoir system to be in the same quasifree, gauge-invariant state ω_R on the CAR algebra, which is determined by its two-point correlations

$$\omega_R(A^*(f)\, A(g)) = \langle Rf, g \rangle_{L^2(I)}$$

where $R\colon L^2(I) \to L^2(I)$ is a positive operator with $0 \leqslant R \leqslant 1$. ω_R is assumed to be invariant under the free dynamics, entailing that R is a multiplication operator, $[Rf](\omega) = r(\omega) f(\omega)$; one can see that $0 \leqslant r(\omega) \leqslant 1$ gives the occupation probability of the mode ω in the state ω_R. The Hamiltonian of the reservoir on the GNS Hilbert space $\mathscr{H}_{\omega_R}$ will be denoted by H_R.

In this section, I take a simple one-particle exchange as the system reservoir interaction. There are two natural choices for the reservoir:

(a) "Individual reservoir": the reservoir consists of N Fermi systems, one for each degree of freedom $a_n^{\#}$, $n = 1,..., N$, in the object system, and one has for the interaction Hamiltonian

$$H_I = \lambda \sum_n a_n^* A_n(f_c) + A_n^*(f_c)\, a_n, \qquad f_c \in L^2(I) \text{ fixed} \tag{24}$$

(b) "Collective reservoir": each degree of freedom is coupled to the same Fermi system which constitutes the reservoir, and

$$H_I = \lambda \sum_n a_n^* A(f_c) + A^*(f_c)\, a_n \tag{25}$$

H_I is a bounded operator on $\mathscr{H}_S \otimes \mathscr{H}_{\omega_R}$, and λ is a coupling constant. The coupling function $f_c(\omega)$ determines the weight of each energy mode ω in the interaction process.

If we now consider an uncorrelated initial state, $\rho \otimes \omega_R$, $\rho \in \mathscr{T}(\mathscr{H}_S)_{+,1}$ (positive trace class operators on $\mathscr{H}_S$ of norm 1), the reduced dynamics of the system $\rho(t)$, defined by

$$\mathrm{Tr}_{\mathscr{H}_S}[\rho(t) A] = \mathrm{Tr}_{\mathscr{H}_S \otimes \mathscr{H}_{\omega_R}}[(e^{-iHt} \rho \otimes \omega_R e^{iHt}) A \otimes 1] \qquad \forall A \in \mathscr{B}(\mathscr{H}_S) \tag{26}$$

with the full Hamiltonian

$$H = H_S + H_R + H_I \tag{27}$$

can be calculated explicitly:

$$\frac{d}{dt} \rho(t) = -i[H_S, \rho](t) + \lambda^2 \int_0^t \mathscr{K}(s)\, \rho(t-s)\, ds \tag{28}$$

The reservoir structure and H_I enter into the integral kernel $\mathscr{K}(s)$ (see ref. 4 for an explicit expression) only via the correlation functions [$A^{\#}(f_c)(t)$ denotes the free reservoir dynamics]

$$\begin{aligned} C_1(t) &= \omega_R(A(f_c)(t)\, A^*(f_c)) \\ &= \langle (1-R) f_c, e^{-i\omega t} f_c \rangle_{L^2(I)} \\ &= \int_I d\omega\, [1 - r(\omega)]\, |f_c(\omega)|^2\, e^{-i\omega t} \end{aligned} \tag{29a}$$

and

$$\begin{aligned} C_2(t) &= \omega_R(A^*(f_c)(t)\, A(f_c)) \\ &= \langle R(e^{-i\omega t} f_c), f_c \rangle_{L^2(I)} \\ &= \int_I d\omega\, r(\omega)\, |f_c(\omega)|^2\, e^{i\omega t} \end{aligned} \tag{29b}$$

Moreover, one has $\mathscr{K}(s) = 0 \Leftrightarrow C_{1/2}(t) = 0$.

If we let τ_R be the decay time of the correlation functions, and τ_S the typical variation time of ρ due to the interaction with the reservoir (i.e., in the interaction picture), it is now easy to see qualitatively under which conditions the memory terms in (28) can be neglected, so that the subdynamics receives a semigroup structure

$$\frac{d}{dt}\rho(t)=\mathscr{L}[\rho(t)]$$

(and hence becomes tractable): we have (in the interaction picture)

$$\int_0^t \mathscr{K}(s)\,\rho(t-s)\,ds \approx \int_0^{\tau_R} \mathscr{K}(s)\,\rho(t-s)\,ds \overset{!}{\approx} \left[\int_0^{\tau_R} \mathscr{K}(s)\,ds\right]\rho(t) \quad (30)$$

if $\rho(t-s)\approx\rho(t)$ in $[0, \tau_R]$, i.e., if $\tau_S \gg \tau_R$. This qualitative argument can be made rigorous in the limit

$$\tau_S/\tau_R \to \infty$$

which can be performed in two ways: the weak coupling limit and the singular coupling limit.

3.1. Weak Coupling Limit

Here one lets $\lambda\to 0 \Rightarrow \tau_S\to\infty$, keeping τ_R constant $(<\infty)$; in order to see the effect of the reservoir, one has to rescale the time: $\tau=\lambda^2 t$. Thus, the price one has to pay for obtaining a semigroup description is the fact that free evolution and the dissipative motion effected by the reservoir are on different time scales: the approximating dynamics is

$$\rho(t)=\exp(-i[H_S,\cdot\,]\,t+K\tau)\rho=\exp([-i[H_S,\cdot\,]+\lambda^2 K]\,t)\rho \quad (31)$$

with [in the case (24)]

$$K\rho=\int_0^\infty dt\sum_n \{C_1(t)[a_n\rho, a_n^*(t)]+C_2(t)[a_n^*\rho, a_n(t)]+\text{h.c.}\} \quad (32)$$

where

$$a_n^\#(t)=e^{iH_S t}a_n e^{-iH_S t}$$

(This is the generator most often used,[10–12] although it can be shown[28] that the physically better choice is an average over K.) For this reason, the weak coupling limit can practically only be used in situations where the free system dynamics plays no role, as in the approach to equilibrium; otherwise, it completely dominates the dissipative part $\lambda^2 K$ for small λ.

K is heavily dependent on H_S and, in the usual case where ω_R is the β-KMS state [i.e., $r(\omega)=e^{-\beta\omega}/(1+e^{-\beta\omega})$], on β: typically, it has the Gibbs state $\exp(-\beta H_S)$ as a global attractor. Thus, the system is still widely determined by its internal structure H_S.

3.2. Singular Coupling Limit

Here, one lets $\tau_R \to 0$, while τ_S is constant ($\lambda = 1$). The most general situation in which this happens is contained in the following easy lemma:

Lemma. $\tau_R \to 0$ if and only if the reservoir is "singular," i.e., if and only if

$$\sigma(h) = I = \mathbb{R}$$

$$R = r1, \quad \text{i.e.,} \quad r(\omega) \equiv r$$

$$f_c \to \text{const}$$

Proof. $\tau_R \to 0$ means $C_{1/2}(t) \to C_{1/2}\delta(t)$. In (29a) and (29b), the δ-function can only be obtained as the Fourier transform of the constant function; therefore, one needs the integration range I to be the whole line, and both $r(\omega)\,|f_c(\omega)|^2$ and $[1-r(\omega)]\,|f_c(\omega)|^2$ to be constant. This implies the assertion. (In the special case $r=\frac{1}{2}(\beta=0)$, one can let $I=[0,\infty)$; in this case, $C_1(t)+C_2(t) \to \delta(t)$ without approximating δ-functions individually.[29]) ∎

If we take $f_c^\varepsilon := g\exp(-\varepsilon\omega^2)$, in the singular coupling limit $\varepsilon \to 0$ we obtain the subdynamics[3]

$$\rho(t) = \exp([-i[H_S, \cdot\,] + \bar{K}]t)\rho \tag{33}$$

with [in the case (24)]

$$\bar{K}\rho = \pi g^2 \sum_n \{(1-r)[a_n\rho, a_n^*] + r[a_n^*\rho, a_n] + \text{h.c.}\} \tag{34}$$

The properties of $\bar{K}$ are very different from those of K: it is neither dependent on H_S nor on a β, since in the allowed states ω_R, $R = r1$, there is no defined temperature (with the above-mentioned exception of $r = 1/2$); therefore, $\bar{K}$ does not describe thermodynamic behavior, i.e., approach to equilibrium. In fact, no such general statement about its effects can be made; note, however, that $\bar{K}$, being of the same strength as H_S, can alter the behavior of the system quite drastically.

The presence of "unusual" states ω_R with $R = r1$ and the fact that the energy spectrum of the reservoir is unbounded from below are the price to pay in this case for the semigroup structure of the dynamics of the open system. Such a price might be considered very high, and higher than in the weak coupling limit. However, the conditions of the lemma can be physically interpreted in the following way. Consider a situation in which the energies of a regular reservoir (with Hamiltonian H_R, state $\omega_R =$ β-KMS state, and $I = [-c, \infty))$ are much bigger than the energies of the system, so that they should be measured on a different scale: $\omega =$ energy scale of the reservoir, $x = \omega/\varepsilon^2 =$ energy scale of the system. On x, the reservoir structures look like

$$\bar{f}_c(x) = f_c(\varepsilon^2 x), \qquad H_R(x) = \frac{1}{\varepsilon^2} H_R$$

$$\bar{r}(x) = r(\varepsilon^2 x), \qquad I(x) = \left[-\frac{c}{\varepsilon^2}, \infty \right)$$

so that the Hamiltonian is (e.g.)

$$H = H_S + \frac{1}{\varepsilon^2} H_R + \sum_n \{a_n^* A_n(\bar{f}_c) + \text{h.c.}\}$$

and the correlation functions look like

$$\begin{aligned} C_2(t) &= \int_{I(x)} dx\, |\bar{f}_c(x)|^2\, \bar{r}(x)\, e^{ixt} \\ &= \int_{-c/\varepsilon^2}^{\infty} dx\, |f_c(\varepsilon^2 x)|^2\, r(\varepsilon^2 x)\, e^{ixt} \\ &\xrightarrow{\varepsilon \to 0} \int_{-\infty}^{\infty} dx\, |f_c(0)|^2\, r(0)\, e^{ixt} = 2\pi\, |f_c(0)|^2\, r(0)\, \delta(t) \end{aligned} \tag{35}$$

In the limit $\varepsilon \to 0$, all the system sees of the reservoir is a neighborhood of the energy mode $\omega = 0$, and there the coupling function f_c and the occupation probabilities are approximately constant, being equal to $f_c(0)$ and $r(0)$. In other words: even though the energy modes of the reservoir and its state are "really" (=on the ω scale) regular, they "look" singular from the point of view of the system (=on the x scale).

[Remark: On the ω scale, the Hamiltonian reads[15]

$$H = \varepsilon^2 H_S + H_R + \varepsilon \sum_n \{a_n^* A_n(f_c) + \text{h.c.}\} \tag{36}$$

and hence the limit $\varepsilon \to 0$ is formally a weak coupling problem. This mathematical equivalence between weak coupling limit and singular coupling limit was first pointed out by Palmer.[15] However, his statement that the distinction between the two "depends on which of two possible time scales is regarded as natural or physical" seems to be somewhat misleading: the "weak coupling limit" with (36) shows none of the typical features of the weak coupling limit proper described above; the sub-dynamics it leads to is

$$\rho(t) = \exp([-iH_S + \bar{K}]\tau), \qquad \tau = \varepsilon^2 t$$

with $\bar{K}$ being the singular coupling generator (34).]

In conclusion, one can say that both coupling limits are characterized by distinct energy proportions: in the weak coupling limit, the coupling energy is very small compared to the system energy and reservoir energy; in the singular coupling limit, interaction and system energies are similar but much smaller than the reservoir energies. On this account, neither one of them is more or less physical than the other—the important question is rather: what are the real energy proportions in the system to be described and do they justify the use of any one of the limits? The physics of the two situations is obviously quite different, which is reflected in the fact that the resulting semigroups have very different properties.

4. THE CURRENT-DRIVEN JUNCTION

4.1. Coupling to External Drive Wires

In this section, we connect the Josephson junction to external wires acting as a dc current source; this models the experimental situation of a current-biased junction (ref. 30, Chapter 6). The wires are metallic systems, taken to be in the normal phase, which drive particles through the junction; of course, we take into account only their physically relevant part, the conduction band, which can be described as a free electron gas.

As mentioned in the introduction, there are various possibilities for the coupling of such wires to the superconductors, due to the options individual/collective reservoirs, or electrons/Cooper pairs to be fed into (drawn from, respectively) R and S. Since heat baths are in general much bigger than the system they cool, for them individual reservoirs seem to be the proper choice[10–12]; the wires, however, are roughly of the same size as the junction, and accordingly I prefer to describe them as a collective reservoir.

As regards the choice of particles, surely Cooper pairs are best adapted to the superconducting situation; in the interaction process

$$r_N^+ A(f)\, A(g) + A^*(f)\, A^*(g)\, r_N^- \tag{37}$$

two electrons in the states f and g leave the wire and show up in the superconductor as part of the condensate, and vice versa. Thus, the "condensation process" (the breaking up of a Cooper pair, respectively) is built into the coupling, and in this case the wires carry only the dc *super*currents in the junction to and away from it. [One might argue that the condensation process should be taken care of by the BCS interaction in the junction Hamiltonian, and therefore regard an interaction process involving only electrons, i.e., of the form (25), as more natural; for this, see Section 4.4.]

The coupling Hamiltonian, then, is built from (37) and reads

$$H_I^{WR}(N) = N^{1/2}\{r_N^+ A_R^{\mathrm{I}}(f_c)\, A_R^{\mathrm{II}}(g_c) + A_R^{\mathrm{I}}(f_c)^*\, A_R^{\mathrm{II}}(g_c)^*\, r_N^-\} \tag{38}$$

where for technical reasons I choose the N dependence as shown (compare ref. 23) and two independent CAR systems I and II constituting the wire. Of course, there is a similar Hamiltonian $H_I^{WS}(N)$ for the other superconductor-wire connection.

The next step is to decide which (if any) coupling limit to use in order to calculate the effect of the wires on the system dynamics, i.e., in order to find the generator of the induced semigroup. The discussion in Section 3 has shown that to this end one has to look at the energy proportions of the total system. Observe first that there is no reason to suppose that the H_I should be particularly weak—the currents driven through the junction by the wires are of the same magnitude as the (ac) currents in the closed junction described by (23). Second, while the superconductor energies $\varepsilon(\mathbf{k})$, $\mathbf{k} \in \Lambda$, range over a few millieV, the conduction band of the wires extends over a few eV. This shows that we are precisely in a situation justifying the use of the singular coupling limit [and *not* of the weak coupling limit used (nonrigorously) in ref. 20 for this problem].

In accordance with the singular coupling limit, we choose $\omega_{\eta_{R(S)}}$, the quasifree, gauge-invariant state with $R = \eta_{R(S)} 1$, as the state of the wire connected to R (S). We then arrive at the following mathematical structures for the total system [$\mathscr{A}_W := \mathrm{CAR}(L^2(R))^{\mathrm{I}} \otimes \mathrm{CAR}(L^2(R))^{\mathrm{II}}$] (see Fig. 1):

Operator algebra

$$\mathscr{A}_W \otimes \mathscr{A}_{\text{system}} \otimes \mathscr{A}_W \tag{39}$$

Hamiltonian

$$H(\text{wire}) + H_I^{WR}(N) + H_{\text{system}}(N) + H_I^{WS}(N) + H(\text{wire}) \tag{40}$$

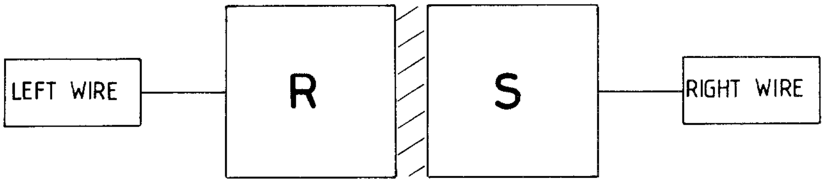

Fig. 1. The structure of the open Josephson junction.

State (at $t=0$)

$$(\omega_{\eta_R} \otimes \omega_{\eta_R}) \otimes \omega_{\text{system}} \otimes (\omega_{\eta_S} \otimes \omega_{\eta_S}) \tag{41}$$

Here, $\eta_S > \eta_R$ models the current source character of the wires, the sign of the current being in accordance with (15) (positive from left to right); we set $\eta_S + \eta_R = 1$ for particle conservation.

These structures will be analyzed in the next section in terms of the subdynamics of the open system, the current driven Josephson junction, in the infinite volume limit.

4.2. Equations of Motion for the Intensive Observables

As outlined in the introduction, we first look at the subdynamics of the finite system. Due to the symmetry of the structures (39)–(41), it suffices to consider only one side, say R with $H_I^{WR}(N)$. Setting $f_c(\omega) = f(0)\exp(-\varepsilon\omega^2)$, we are in essence in the range of application of the theorem proved in ref. 3; the only difference is that, due to the two-particle nature of (38), the relevant correlation functions are the products

$$D_i(t) = C_i^{\text{I}}(t)\, C_i^{\text{II}}(t), \qquad i = 1, 2$$

for which [compare (29a), (29b)]

$$\begin{aligned}
\lim_{\varepsilon \to 0} D_1(t) &= 2\pi(1-\eta_R)^2\, |f_c(0)|^2\, \|g_c\|_{L^2}^2\, \delta(t) \\
&=: \gamma(1-\eta_R)^2\, \delta(t) \\
\lim_{\varepsilon \to 0} D_2(t) &= 2\pi(\eta_R)^2\, |f_c(0)|^2\, \|g_c\|_{L^2}^2\, \delta(t) \\
&=: \gamma\eta_R^2 \delta(t)
\end{aligned}$$

Then ref. 3 shows the existence of the singular coupling limit $\varepsilon \to 0$;

specializing the generator of the induced semigroup on $\mathscr{A}_R(N)$ [(3)] to our case, we obtain in the Heisenberg picture

$$G_R(N)(A)=\frac{\gamma}{2}N\{(1-\eta_R)^2[r_N^+,A]\,r_N^- +\eta_R^2[r_N^-,A]\,r_N^+ +\text{h.c.}\} \quad (42)$$

for all self-adjoint $A\in\mathscr{A}_R(N)$ [compare (34)].

Similarly, one gets for the effect of the right wire on S

$$G_S(N)(A)=\frac{\gamma}{2}N\{(1-\eta_S)^2[s_N^+,A]\,s_N^- +\eta_S^2[s_N^-,A]\,s_N^+ +\text{h.c.}\} \quad (43)$$

for all self-adjoint $A\in\mathscr{A}_S(N)$.

The total generator is then

$$G_N:=G_R(N)+G_S(N) \quad (44)$$

so that the full dynamics for the finite system is given by

$$\frac{d}{dt}A=i[H_{\text{system}}(N),A]+G_N(A) \qquad \forall A\in\mathscr{A}_{\text{system}}(N) \quad (45)$$

This yields as equations of motion for the physically relevant intensive observables (16) [compare (17)]

$$\begin{aligned}
\frac{d}{dt}r_N^3&=j_N-\gamma(\eta_S-\eta_R)\,c_N^R\\
\frac{d}{dt}s_N^3&=-j_N+\gamma(\eta_S-\eta_R)\,c_N^S\\
\frac{d}{dt}c_N^R&=-2j_N r_N^3+2\gamma(\eta_S-\eta_R)\,c_N^R r_N^3\\
\frac{d}{dt}c_N^S&=2j_N s_N^3-2\gamma(\eta_S-\eta_R)\,c_N^S s_N^3\\
\frac{d}{dt}w_N&=-2\Delta\varepsilon\, j_N-(4K+4g)\,j_N z_N+2\gamma(\eta_S-\eta_R)\,w_N z_N\\
\frac{d}{dt}j_N&=2\Delta\varepsilon\, w_N+(4K+4g)\,w_N z_N-4\lambda^2(c_N^R s_N^3-c_N^S r_N^3)+2\gamma(\eta_S-\eta_R)\,j_N z_N
\end{aligned} \quad (46)_N$$

Note here that due to the same basic interaction ansatz (37), structurally equivalent reservoir terms have been obtained in a different (nonrigorous) way in refs. 24 and 25.

In order to determine now the dynamics of the expectation values for the states $\omega_{\text{system}} =: \omega \in \partial S^p(\mathscr{A})$ and in the thermodynamic limit $N \to \infty$, we need an extension of Hepp and Lieb's theorem used in Section 2.2 from automorphisms to contraction semigroups of the type (45). This is, fortunately, not very difficult and has been elaborated in ref. 22, Theorem I. The only assumption for the proof to go through is that the corresponding classical system of differential equations $[(46)_N$ without N indices $=: (46)]$ has global solutions for initial values corresponding to $\omega \in \partial S^p$:

$$(r^3(0),..., j(0)) = (\lim_{N \to \infty} \omega(r^3_N),..., \lim_{N \to \infty} \omega(j_N)), \qquad \omega \in S^p(\mathscr{A}) \tag{47}$$

This is contained in the following proposition, which in its general form supplements Theorem I of ref. 22 and will be proven in the Appendix:

Proposition. The compact set $D \subseteq \mathbb{R}^6$ of initial conditions (47) is positive invariant under the classical flow defined by (46).

The preceding discussion has shown that (46) gives the full dynamics of the physical variables of the open system in the thermodynamic limit. In this paper, I only analyze these equations under the assumptions (18), making it possible to rewrite them in the variables (22):

$$\frac{d}{dt} c(t) = -2z(j(t) - \gamma(\eta_S - \eta_R)\, c(t)) \Leftrightarrow \frac{d}{dt} [c(t) + z^2(t)] = 0 \tag{48a}$$

$$\begin{aligned} \frac{d}{dt} z(t) &= j(t) - \gamma(\eta_S - \eta_R)\, c(t) \\ &= 2\lambda c(t) \sin \Delta\varphi(t) - \gamma(\eta_S - \eta_R)\, c(t) \end{aligned} \tag{48b}$$

$$\frac{d}{dt} \Delta\varphi(t) = 2\Delta\varepsilon + (4K + 4g)\, z(t) - 4\lambda \cos \Delta\varphi(t)\, z(t) \tag{48c}$$

These are the final dynamical equations for the macroscopic behavior of the current-driven Josephson junction. The physical region of R^3 is here

$$D = \{c, z, \Delta\varphi \mid c \in [0, 1/4], z \in [-1/2, 1/2], \Delta\varphi \in [0, 2\pi]\}$$

This can be seen using (4) and the product structure of the states $\omega \in \partial S^p(\mathscr{A})$.

4.3. The dc Josephson Effect

In the macroscopic dynamics (48) the effect of the external drive wires on the dynamics of the closed junction (23) is solely to add the term

$$j_{\text{ex}}(t) := \gamma(\eta_S - \eta_R)\, c(t) \tag{49}$$

which clearly describes the external current. As was to be expected from the microscopic model structures, the magnitude of this current depends on the "current source strength" $\eta_S - \eta_R$; the dependence on the density of the condensed pairs reflects the fact that only supercurrents are included in the present description.

If there is no dc voltage across the junction, $\Delta\varepsilon = 0$, one has the following unique stationary solution of (48) in the physical domain D (if one excludes the case $c \equiv 0$—no superconducting phase—and observes[21] $\lambda \ll K, g$):

$$z_0 = 0, \qquad c_0 = c(0)$$

$$\sin \Delta\varphi_0 = \frac{\gamma(\eta_S - \eta_R)}{2\lambda}, \qquad \cos \Delta\varphi_0 = \pm(1 - \sin^2 \Delta\varphi_0)^{1/2} \tag{50}$$

$$j = j_{\mathrm{ex}} = \gamma(\eta_S - \eta_R)\, c_0 \qquad \text{with} \quad \gamma(\eta_S - \eta_R) \in [0, 2\lambda] \tag{51}$$

This is just the dc Josephson effect. Condition (51) reflects the fact that j_{ex} must not be greater than the critical current $j_0 = 2\lambda c_0$. The free parameter c_0 of the solutions is determined physically by the temperature of the junction. The question whether $\cos \Delta\varphi_0 < 0$ or $\cos \Delta\varphi_0 > 0$ gives the physically correct solution can be settled[30] by arguing that the barrier energy $w = 2\lambda c_0 \cos \Delta\varphi_0$ should be minimal, hence $\cos \Delta\varphi_0 < 0$.

Finally, if one neglects the small correction (48a) to the usual assumption $c \equiv \mathrm{const}$ (compare ref. 21) and the equally small last term in (48c), one obtains the system

$$\frac{d}{dt} z(t) = 2\lambda c_0 \sin \Delta\varphi(t) - \gamma(\eta_S - \eta_R)\, c_0 \tag{52a}$$

$$\frac{d}{dt} \Delta\varphi(t) = (4K + 4g)\, z(t) \tag{52b}$$

which is structurally equivalent to the phenomenological model of Anderson[31]: (52) is a Hamiltonian system with Hamilton function

$$H = (2K + 2g)\, z^2 + V(\Delta\varphi)$$

$$V(\Delta\varphi) = 2\lambda c_0 \cos \Delta\varphi + \gamma(\eta_S - \eta_R)\, c_0\, \Delta\varphi$$

with $V(\Delta\varphi)$ being the famous "washboard" potential.

Thus, the phenomenological theories can be fully recovered from the microscopic treatment presented here.

4. ALTERNATIVE MODELS

It might seem almost trivial that a coupling of the Josephson junction to external wires should yield a satisfactory description of the dc Josephson effect by the ensuing macroscopic dynamical equations, as is indeed the case for the model developed above. This is, however, *not* true, as can be seen explicitly by considering alternative models for the wires and their interaction with the superconductors.

Let us look at the model sketched in ref. 1. Here, individual reservoirs were taken, i.e., the operator algebra for the wire is

$$\mathscr{A}_W = \bigotimes_{n \in \mathbb{N}} \bigotimes_{\sigma = +,-} \mathrm{CAR}(L^2(R))_{\sigma n} \tag{53}$$

so that there is one infinite CAR system for each degree of freedom $+n = (\mathbf{k}, \uparrow)$, $-n = (-\mathbf{k}, \downarrow)$. The interaction is chosen to be a single-electron exchange

$$H_I^{WR}(N) = \sum_n \sum_{\sigma = +,-} c_{\sigma n}^{R+} A_{\sigma n}(f) + A_{\sigma n}(f)^* \, c_{\sigma n}^{R-} \tag{54}$$

where the $c_{\sigma n}^{\#}$ are the usual electron creation and annihilation operators for the superconductor R. Inserting this into the structures (39)–(41), the singular coupling limit yields the generator (44) with

$$G_R(N)(A) = \frac{\gamma}{2} \sum_n \sum_{\sigma = +,-} \{(1 - \eta_R)[c_{\sigma n}^{R+}, A] \, c_{\sigma n}^{R-} + \eta_R [c_{\sigma n}^{R-}, A] \, c_{\sigma n}^{R+} + \text{h.c.}\} \tag{55}$$

One easily checks that G_N gives the same reservoir terms in the dynamical equations for the intensive observables as Hepp and Lieb obtained with their methods (which, however, require setting $\eta_R = 0$, $\eta_S = 1$). Proceeding as before, one arrives at the following dynamical description of the junction, taking the place of (48):

$$\frac{d}{dt} c(t) = -2j(t)\, z(t) - 2\gamma c(t)$$

$$\frac{d}{dt} z(t) = j(t) - \gamma z(t) - \frac{1}{2} \gamma (\eta_S - \eta_R) \tag{56}$$

$$\frac{d}{dt} \Delta\varphi(t) = 2\Delta\varepsilon + (4K + 4g)\, z(t) - 4\lambda \cos \Delta\varphi(t)\, z(t)$$

In search for the dc Josephson effect, we set $\Delta\varepsilon = 0$ and look for stationary solutions of (56) in the physical domain D, observing as before that $c \neq 0$ and $\lambda \ll K$, g. There are none!

Stationary solutions exist only under the unphysical condition $\lambda \geq K + g$:

$$c_0 = \frac{1}{4}\frac{\gamma}{\Gamma}\left[(\eta_S - \eta_R) - \frac{\gamma}{\Gamma}\right], \qquad z_0 = -\frac{1}{2}\frac{\gamma}{\Gamma}$$

$$\cos\Delta\varphi_0 = \frac{K+g}{\lambda}, \qquad \sin\Delta\varphi_0 = \frac{\Gamma}{\lambda} \tag{57}$$

$$j = \gamma z_0 + \frac{\gamma}{2}(\eta_S - \eta_R) = \frac{1}{2}\gamma\left[(\eta_S - \eta_R) - \frac{\gamma}{\Gamma}\right]$$

with $\Gamma := [\lambda^2 - (K+g)^2]^{1/2}$ and $\eta_S - \eta_R \in [\gamma/\Gamma, 1]$, $\gamma \leq \Gamma$.

Even if we were to forget about the fact that $\lambda \ll K + g$ in the real system, this solution could not be interpreted as the dc Josephson effect: since $z \neq 0$, we have [with (13)] $V \neq 0$, which is physically wrong. Furthermore, it is quite strange that c_0 (and therefore the critical current $j_0 = 2\lambda c_0$) should depend on the current source strength.

It is not easy to interpret this unphysical behavior of the model (53), (54) (one can show with the methods of ref. 14 that it is not due to the limit procedure of the singular coupling limit). Our best guess is that the wires (53) are "too big" for the system: for realistic parameters λ, K, g the only way to keep the phase difference constant (so as to have a dc Josephson current j) is the physically adequate condition $V = 0$, $\Rightarrow z = 0$; but in this case the reservoir destroys the superconducting phase via $(d/dt)c = -2\gamma c$.

The same problem occurs if we connect this current source to a *single* superconductor: in this case, we have

$$\frac{d}{dt}c = 0, \qquad \frac{d}{dt}r^3 = 0, \qquad \frac{d}{dt}\varphi_R = 2\varepsilon_R + 2gr^3$$

for the closed and

$$\frac{d}{dt}c = -4\gamma c, \qquad \frac{d}{dt}r^3 = -2\gamma r^3, \qquad \frac{d}{dt}\varphi_R = 2\varepsilon_R + 2gr^3$$

for the open system. Again, the superconductivity is destroyed by the wires. An analogous argument applies to the model in ref. 26, where the same wire structures were used to carry Fiske currents. One can look at this phenomenon as an example of the capability of the reservoir to alter internal system structures, mentioned in Section 3.

Unfortunately, we cannot let electrons flow out of a collective reservoir, since the corresponding model transcends fundamentally the mathematical framework in which we work: the intensive observables

$$c_{\sigma N}^{R\pm} := \frac{1}{N} \sum_n c_{\sigma N}^{R\pm}, \qquad \sigma = +, -$$

appearing in that model no longer have $O(1/N)$ commutators, a basic prerequisite for the entire reasoning of ref. 1 followed here.

Thus, while, due to the view regarding an electron interaction between wire and superconductor as natural, we would consider it interesting to have a successful such model, there seems to be none in sight.

I have, however, constructed yet other models with individual reservoirs, which I shall not discuss here for reasons of space; they confirm what I hope the discussion of this section has shown: in situations where it is physically appropriate to use the singular coupling limit for the study of many-body open systems, the qualitative behavior of the macroscopic dynamical equations depends sensitively on the coupling constants [see γ in (57)], on the nature of the coupling, and on the size (relative to the object system) of the reservoir.

APPENDIX

In the notation of Refs. 1 and 22, the intensive observables (which I assume to be self-adjoint here) averaging over the subsystems n are denoted by α_N^k, $k = 1,..., L$; they obey the quantum equations

$$\frac{d}{dt} \alpha_N^k(t) = i[H_{\text{system}}(N), \alpha_N^k](t) + G_N(\alpha_N^k)(t) =: f(\alpha_N^1(t),..., \alpha_N^L(t)) \qquad \text{(A1)}$$

where f is a polynomial. The corresponding classical differential equations [obtained formally by dropping the N indices in (A1)],

$$\frac{\partial}{\partial t} \underline{\alpha}(t, \underline{\gamma}) = f(\underline{\alpha}(t, \underline{\gamma})), \qquad \underline{\alpha}(0, \underline{\gamma}) = \underline{\gamma} \qquad \text{(A2)}$$

define a flow $\underline{\alpha}(t, \underline{\gamma})$ on $\mathbb{R}^L$.

In this general setting, I prove the following.

Proposition. 1. Solutions of (A2) which start in the compact set

$$D := \{\underline{\gamma} \in R^L \mid \underline{\gamma} = \lim_{N \to \infty} \omega(\underline{\alpha}_N) \text{ for an } \omega \in \partial S^p\} \subseteq \mathbb{R}^L \qquad \text{(A3)}$$

are defined for all $t \geqslant 0$ and are bounded by

$$c := \max_k \left(\lim_N \|\alpha_N^k\|\right) < \infty$$

2. Theorem I of ref. 22 holds for $\omega \in \partial S^p$; i.e.,

$$\lim_N \omega(\alpha_N^k(t)) = \alpha^k(t, \underline{\gamma}) \qquad \forall t \geqslant 0, \quad \forall k = 1,..., L, \quad \forall \omega \in \partial S^p$$

In particular, ∂S^p is positive invariant and hence D is positive invariant under the flow $\underline{\alpha}(t, \underline{\gamma})$.

Proof. 1. Assume that a solution with initial value $\underline{\gamma}_0 \in D$ corresponding to an $\omega_0 \in \partial S^p$ is maximally defined on a finite interval $[0, b)$. Then one can find[32] a $0 \neq \bar{b} < \bar{b} + \varepsilon < b$, ε small, such that

$$|\alpha^k(\bar{b}, \underline{\gamma}_0)| > c \qquad \text{for a } k \in \{1,..., L\} \tag{A4}$$

Without restricting generality, one can suppose that

$$|\underline{\alpha}(t, \underline{\gamma}_0)| \leqslant |\underline{\alpha}(\bar{b}, \underline{\gamma}_0)| \qquad \forall t \leqslant \bar{b}$$

One can then follow the Picard iteration scheme for (A2), precisely as in refs. 1 and 22, for intervals $[ld_2, (l+1)\, d_2)$, $l = 0, 1,...,$ until one gets to the interval with $ld_2 \leqslant \bar{b} + \varepsilon \leqslant (l+1)\, d_2$; here we use the iteration scheme up to $\bar{b} + \varepsilon$. The arguments of the mentioned references show that on the interval $[0, \bar{b} + \varepsilon]$, and thus in particular for $t = \bar{b}$, it holds that

$$\lim_N \omega_0(\alpha_N^k(t)) = \alpha^k(t, \underline{\gamma}_0) \qquad \forall k = 1,..., L \tag{A5}$$

But this entails

$$|\alpha^k(\bar{b}, \underline{\gamma}_0)| = \lim_N |\omega_0(\alpha_N^k(\bar{b}))| \leqslant \lim_N \|\alpha_N^k(\bar{b})\| \overset{*}{\leqslant} \lim_N \|\alpha_N^k\| \leqslant c \qquad \forall k$$

which contradicts (A4). The starred inequality is true because of the contraction property of the semigroup defined by (A1).[8]

2. Part 1 shows that Theorem I of ref. 22 is true for $\omega \in \partial S^p$, and in fact for all pure classical states in the sense of ref. 1 (we have not used the permutation invariance of the states). In particular, (A5) holds for all $t \geqslant 0$, so that pure classical states remain pure for all times if one looks at the dynamics in the Schrödinger picture. Since the generator in (A1) is permutation invariant, this property of the states is also conserved in the course of time, so that one can conclude the positive invariance of ∂S^p. This implies the positive invariance of D under the flow $\underline{\alpha}(t, \underline{\gamma})$.

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